Geometric Modeling
Assignment sheet #11
"Splines & Differential Geometry"
(due July 17th 2012 before the lecture)

Silke Jansen, Ruxandra Lasowski,
Art Tevs, Michael Wand

Exercise 1 (De Casteljau Algorithm for Bézier Triangles): **[3 points]**

A quadratic Bézier triangle is given by the control points

$$F(a,a)=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad F(a,b)=\begin{pmatrix}2\\2\\4\end{pmatrix}\quad F(a,c)=\begin{pmatrix}4\\-2\\6\end{pmatrix}$$

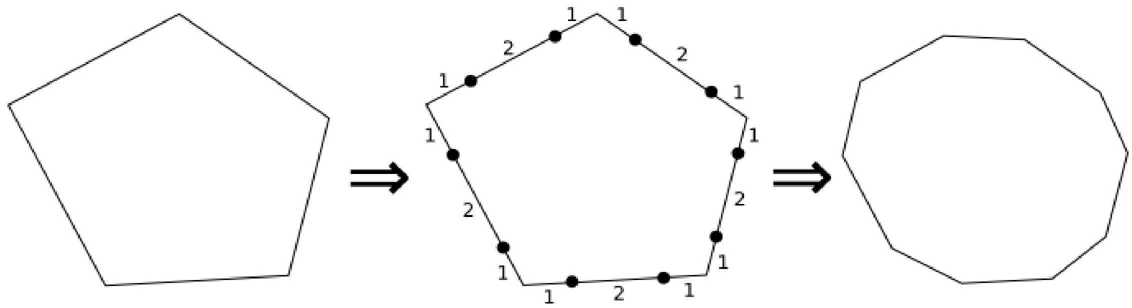
$$F(b,b)=\begin{pmatrix}4\\4\\0\end{pmatrix}\quad F(b,c)=\begin{pmatrix}8\\0\\4\end{pmatrix}\quad F(c,c)=\begin{pmatrix}6\\-4\\4\end{pmatrix}$$

at parameter positions a=(0,0), b=(1,0), c=(0.5,1).

Which of the parameter pairs p_1=(0.25,0.5), p_2=(0.3,0.75), p_3=(0.5,0.5) is outside the triangle? For the other parameter pairs p, evaluate the surface F(p,p) using the De Casteljau Algorithm for Bézier triangles.

Exercise 2 (Chaikin's Corner Cutting): **[5 points]**

Consider a closed control polygon. Chaikin's algorithm can be formulated as subdividing each linear segment 1:2:1 and using the arising points as the control points of the refined control polygon:



Show that the limit curve is a C1-continuous, piecewise quadratic Bézier curve.

Hint: Remember the De Casteljau algorithm.

Exercise 3 (Intrinsic Mapping): **[2+1 points]**

Consider the mapping function $f(u,v)=\begin{bmatrix}u\cos(v)\\u\sin(v)\\v\end{bmatrix}$ from a 2D domain to a surface.

a. Find the angle between two intersecting curves $u+v=0$ and $u-v=0$ on the surface.
b. What kind of mapping function does $f(u,v)$ belong to (conformal, equiareal, isometric)?

Exercise 4 (Euler Curvature Formula): **[3 point]**

Let $k_n(\varphi) = k_{\max} \cos^2(\varphi) + k_{\min} \sin^2(\varphi)$ be the curvature associated with the angle between the current tangent and the tangent of the maximal principal curvature.

Proof that the mean curvature is actually the mean curvature:

$$H = \frac{1}{2\pi} \int_0^{2\pi} k_n(\varphi) d\varphi$$

i.e.: The mean of all directional curvatures is the mean of the principal curvatures ($k_{max} + k_{min}$)/2.

Exercise 5 (Ruled and Developable Surfaces): **[2+2+3 points]**

Definition: Ruled surface

A *ruled surface* can be described (at least locally) as the set of points swept by a moving straight line.

Let $p(t)$ be a given space curve and $a(t)$ be a unit vector of the sweeping line. Then the ruled surface $r(u,v)$ is given as :

$$r(u,v) = p(u) + va(u) \qquad \|a(u)\| = 1$$

The Gaussian curvature on a ruled surface is ≤ 0 everywhere.

a. Is the surface given by $z = xy + 3$ a ruled surface? Justify your answer.

b. Present a surface which is a ruled surface *and* a surface of revolution at the same time. Justify your answer.

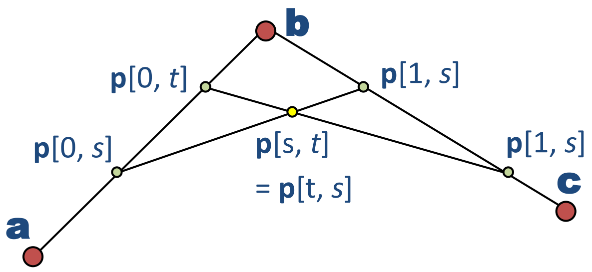
c. Given a space curve $r(s)$ parameterized by its arc-length s, consider a surface generated by all the tangents to the curve. Is it a developable surface? Justify your answer.

Hint: Write down the surface equation as a ruled surface and determine the Gaussian curvature using fundamental forms.

“Bonus Exercises” (due July 18th 2012 before the tutorial)

The following exercises are optional. Points obtained will be added to the overall assignment score.

Bonus Exercise 1 (Why Blossoms are symmetric): [4 points]



Consider a de Casteljau scheme as shown in the figure above. Prove that the following two operations yield the same result:

1. Affine interpolation between **a** and **b** with ratios *s*, (1 – *s*) to get “**p**[0, *s*]” and between **b**, **c** with ratios *s*, (1 – *s*) to get “**p**[1, *s*]”. Then interpolate again between the resulting points with ratios t, (1 – *t*) to obtain the point “**p**[*s, t*]”.
2. Affine interpolation between **a** and **b** with ratios *t*, (1 – *t*) to get “**p**[0, *t*]” and between **b**, **c** with ratios *t*, (1 – *t*) to get “**p**[1, *t*]”. Then interpolate again between the resulting points with ratios s, (1 – s) to obtain the point “**p**[*t, s*]”.

In other words, show that the points we have labeled p[s,t] and p[t,s] are identical. Use only the properties of affine interpolation between points in $\mathbb{R}^n$ (we do not yet know that this corresponds to Blossoms). This result is known as Menelaos’ theorem.

Bonus Exercise 2 (Fundamental Forms and Curvature): [1+1 points]

a. Find the coefficients of the second fundamental form of the surface

$$r(u,v) = \begin{pmatrix} \cos v - u\sin v \\ \sin v + u\cos v \\ u + v \end{pmatrix}$$

b. Find the Gaussian curvature of the surface

$$r(u,v) = \begin{pmatrix} -u \\ 2\sin v \\ 2\cos v \end{pmatrix}$$

Bonus Exercise 3 (Principle Curvatures): [2 points]

Show that the principal curvatures of the surface $r(u,v) = \begin{bmatrix} u\cos(v) \\ u\sin(v) \\ e^v \end{bmatrix}$ have opposite signs.